

# DERIVED CATEGORIES OF SHEAVES ON QUASI-PROJECTIVE SCHEMES

MATTHEW ROBERT BALLARD

**ABSTRACT.** We prove that the bounded derived category of coherent sheaves with proper support is equivalent to the category of locally-finite, cohomological functors on the perfect derived category of a quasi-projective scheme over a field. We introduce the notions of pseudo-adjoints and Rouquier functors and study them. As an application of these ideas and results, we extend the reconstruction result of Bondal and Orlov to Gorenstein projective varieties.

## 1. INTRODUCTION

Understanding the geometry of schemes through their derived categories of sheaves has become a popular topic in the past couple of decades. However, much of the results have focused on the case when the scheme in question is smooth. Let us recall two results most relevant to those in this paper. In [5], Bondal and van den Bergh proved that the bounded derived category of coherent sheaves,  $D_{\text{coh}}^b(X)$ , on a smooth and proper variety,  $X$ , is saturated. Meaning, any covariant or contravariant functor,  $\phi : D_{\text{coh}}^b(X) \rightarrow \text{Vect}_k$ , satisfying an appropriate boundedness condition, called local-finiteness, is representable. Moreover,  $D_{\text{coh}}^b(X)$  is equivalent to either of these categories of functors. For the second result, we take two smooth, projective varieties,  $X$  and  $Y$ , and assume that  $\omega_X$  is ample or anti-ample. In [4], Bondal and Orlov proved that, if  $X$  and  $Y$  have equivalent derived categories, then they must be isomorphic.

In either of these cases, removing the assumption of smoothness sabotages the proofs. For a general projective scheme,  $X$ , one can define two different categories that reduce to  $D_{\text{coh}}^b(X)$  if  $X$  is regular. One is the bounded derived category of coherent sheaves and the other is the smallest triangulated subcategory of  $D_{\text{coh}}^b(X)$  containing all finite-rank locally-free sheaves,  $D_{\text{perf}}(X)$ . The first step in trying to extend these results to a general projective scheme is deciding which category will be the focus of the investigation. The main result of this paper, found in section 3, tells us these two categories are very closely related.  $D_{\text{coh}}^b(X)$  is equivalent to the category of locally-finite, cohomological functors on  $D_{\text{perf}}(X)$ . One can also prove that, if the base field is perfect, then  $D_{\text{perf}}(X)$  is equivalent to category of locally-finite, homological functors on  $D_{\text{coh}}^b(X)$ . We approach this result from the perspective of compactly-generated triangulated categories. Proving an equivalence of categories means finding a functor and checking its fullness, faithfulness, and essential surjectivity. The functor we have in the result is a composition of the Yoneda functor and restriction. Its essential surjectivity was established previously in an appendix to [5]. Fullness follows from the same argument as in [5]. Faithfulness is new. However, the main new addition to known results appears to be perspective. Discovering an equivalence of categories allows access to greater set of tools

to tackle other problems. Some simple ideas and corollaries are collected in section 4.

Another application of these ideas is an extension of Bondal and Orlov's result on reconstruction to the case where  $X$  and  $Y$  are Gorenstein. To realize this extension, we need another idea, which is presented in section 5. The idea is a relativization of the notion of a Serre functor. In honor of Rouquier's paper [12], we name this relativization a Rouquier functor. With the Rouquier functor playing the role of the Serre functor in the proof and with the ideas of the preceding, we prove that, if  $X$  is a projective Gorenstein variety with ample or anti-ample canonical bundle and  $Y$  is another projective variety with an equivalent perfect derived category (or bounded derived category of coherent sheaves), then  $X$  and  $Y$  are isomorphic.

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## 2. PRELIMINARIES

First, a note about notational conventions in this paper. For any category  $\mathcal{C}$ , the morphism set between objects  $A$  and  $B$  is denoted as  $[A, B]$ . All functors are covariant.

Before we dive into the bulk of the paper, we will recall, without offering proofs, some ideas and results essential to the proceeding discussion.

Let  $\mathcal{T}$  be a triangulated category possessing all set indexed coproducts. We say an object  $C$  of  $\mathcal{T}$  is **compact** (or small) if, for all collections,  $X_i$ ,  $i \in I$ , of objects in  $\mathcal{T}$ , the natural map

$$\bigoplus_{i \in I} [C, X_i] \rightarrow [C, \coprod_{i \in I} X_i]$$

is a isomorphism. The subcategory of compact objects,  $\mathcal{T}^c$ , is triangulated and closed under taking direct summands.

Given a subcategory  $\mathcal{S}$  of  $\mathcal{T}$ , let  $\mathcal{S}^\perp$  denote the subcategory of objects,  $A$ , so that  $[S, A]$  is zero for all objects,  $S$ , of  $\mathcal{S}$ . We say that  $\mathcal{T}$  is **compactly-generated** if  $(\mathcal{T}^c)^\perp$  is zero.

If  $\mathcal{T}$  is compactly-generated, we have a general criteria for representability of functors from  $\mathcal{T}$  to abelian groups.

**Theorem 2.1.** (*Brown Representability*) *Let  $\mathcal{T}$  be a compactly-generated triangulated category and let  $H : \mathcal{T}^\circ \rightarrow \text{Mod } \mathbf{Z}$  be a functor that converts coproducts to products and sends triangles to long exact sequences.  $H$  is representable.*

For a proof, see [10].

The main example of a compactly-generated triangulated category is the unbounded derived category,  $D(X)$ , of quasi-coherent sheaves on a quasi-compact and separated scheme,  $X$ . One can use Brown representability to efficiently study Grothendieck duality for quasi-compact and separated schemes, see [9].

The category of compact objects of  $D(X)$  admits a more geometric characterization. We say that a complex,  $E$ , in  $D(X)$  is **perfect** if it locally (in the Zariski topology) quasi-isomorphic to a bounded complex of finite-rank locally-free sheaves. If  $X$  possesses the resolution property, i.e. has enough locally-free sheaves, then a complex is perfect if and only if it is quasi-isomorphic to a bounded complex

of finite-rank locally-free sheaves. The subcategory of perfect objects is called the perfect derived category of  $X$  and denoted by  $D_{\text{perf}}(X)$ .

### 3. LOCALLY-FINITE COHOMOLOGICAL FUNCTORS ON THE PERFECT DERIVED CATEGORY

Unless otherwise indicated,  $X$  is a quasi-projective scheme over a field  $k$  and all categories mentioned are  $k$ -linear and triangulated.  $\text{Vect}_k$  denotes the category of vector spaces over  $k$  and  $\text{vect}_k$  denotes the category of finite-dimensional vector spaces over  $k$ .

**Definition 3.1.** Let  $\mathcal{T}$  be a  $k$ -linear triangulated category. A functor  $H : \mathcal{T}^\circ \rightarrow \text{Vect}_k$  is called **cohomological**, if for each exact triangle

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ & \searrow \scriptstyle [1] & \swarrow \\ & C & \end{array}$$

the sequence

$$\cdots \longleftarrow H(C[i-1]) \longleftarrow H(A[i]) \longleftarrow H(B[i]) \longleftarrow H(C[i]) \longleftarrow H(A[i+1]) \longleftarrow \cdots$$

is exact. A functor  $H : \mathcal{T} \rightarrow \text{Vect}_k$  is called **homological**, if for each exact triangle

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ & \searrow \scriptstyle [1] & \swarrow \\ & C & \end{array}$$

the sequence

$$\cdots \longrightarrow H(C[i-1]) \longrightarrow H(A[i]) \longrightarrow H(B[i]) \longrightarrow H(C[i]) \longrightarrow H(A[i+1]) \longrightarrow \cdots$$

is exact.

**Definition 3.2.** Let  $\mathcal{T}$  be a  $k$ -linear triangulated category. A functor  $\phi : \mathcal{T} \rightarrow \text{Vect}_k$  (or  $\phi : \mathcal{T}^\circ \rightarrow \text{Vect}_k$ ) is called **locally-finite** when it satisfies the following condition:

$$\dim_k \left( \bigoplus_{j \in \mathbf{Z}} \phi(A[j]) \right) < \infty \text{ for all } A \in \mathcal{T}.$$

Given a  $k$ -linear triangulated category  $\mathcal{T}$ , we denote by  $\mathcal{T}^\vee$  the category of locally-finite cohomological functors on  $\mathcal{T}$ . We denote by  ${}^\vee\mathcal{T}$  the category of locally-finite homological functors on  $\mathcal{T}$ .

The main result of this section is the following.

**Theorem 3.3.** Let  $X$  be a quasi-projective scheme over a field  $k$ . Then the restricted Yoneda functor provides an equivalence  $D_{\text{coh},c}^b(X) \rightarrow D_{\text{perf}}(X)^\vee$ .

The proof will be accomplished through the following series of lemmas. The restricted Yoneda functor will be defined in the process of the proof.

**Lemma 3.4.** Let  $\mathcal{T}$  be a compactly-generated triangulated category. Any cohomological functor  $F : (\mathcal{T}^c)^\circ \rightarrow \text{vect}_k$  is representable by an object of  $\mathcal{T}$ . Any natural transformation between such functors is induced by a morphism of their representing objects.

*Proof.* We follow [6]. Given such a functor  $F : (\mathcal{T}^c)^\circ \rightarrow \text{Vect}_k$ , we let  $D$  denote the dualization functor on  $\text{Vect}_k$ , i.e.  $D(V) := V^*$ . Then, there is a canonical extension of  $D \circ F$  to a functor  $G : \mathcal{T} \rightarrow \text{Vect}_k$  called the Kan extension. We take  $G(X)$  to be the colimit of  $G(C)$  over all maps from compact objects  $C \rightarrow X$ .

We claim that  $G : \mathcal{T}^\circ \rightarrow \text{Vect}_k$  is homological and takes coproducts to products. For the first statement, let

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ & \swarrow w & \searrow v \\ & Z & \end{array}$$

be a triangle. Composing with  $u, v, w$  gives a sequences of maps

$$\cdots \rightarrow \{C \rightarrow X\} \rightarrow \{C \rightarrow Y\} \rightarrow \{C \rightarrow Z\} \rightarrow \{C \rightarrow X[1]\} \rightarrow \cdots$$

These give a sequence of maps

$$\cdots \rightarrow G(X) \rightarrow G(Y) \rightarrow G(Z) \rightarrow G(X[1]) \rightarrow \cdots$$

It is clear that this is a chain complex. To check that it is exact, we note that any element,  $\xi$ , of  $G(Y)$  can be represented by an element of some  $G(C)$ . If  $\xi$  is zero in  $G(Z)$ , we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ & & \uparrow f & & \uparrow g & & \\ & & C & \xrightarrow{\phi} & D & & \end{array}$$

with  $G(\phi)(\xi) = 0$ . We can complete this to a map of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \uparrow h & & \uparrow f & & \uparrow g & & \uparrow h[1] \\ E & \xrightarrow{\psi} & C & \xrightarrow{\phi} & D & \xrightarrow{\nu} & E[1] \end{array}$$

Since  $G$  is exact on  $\mathcal{T}^c$ , if  $G(\phi)(\xi) = 0$  for  $\xi \in G(C)$  then  $G(\psi)(\mu) = \xi$  for some  $\mu \in G(E)$ . Thus,  $G$  is homological. Since we are mapping out of compact objects, for fixed  $C$ , we have a bijection

$$\{C \rightarrow \coprod_{i \in I} X_i\} = \coprod_{i \in I} \{C \rightarrow X_i\}$$

We have a natural map  $\coprod G(X_i) \rightarrow G(\coprod X_i)$  and the previous identification of sets gives the inverse. Thus,  $G$  commutes with coproducts and the claim is proven.

Applying  $D$  again, we can use Brown representability to deduce that  $D \circ G$  is representable by an object  $A$  of  $\mathcal{T}$ . The restriction of  $D \circ G$  to  $\mathcal{T}^c$  is isomorphic to  $D \circ D \circ F$ . Since  $F$  lands in  $\text{vect}_k$ ,  $D^2$  cancels out and  $A$  is the object we seek.

Given a natural transformation between two such functors, we can repeat the argument and use the Yoneda lemma to extract the appropriate morphism between the representing objects.  $\square$

We can rephrase this as follows. The inclusion  $\mathcal{T}^c \hookrightarrow \mathcal{T}$  induces a restricted Yoneda functor.

$$\mathcal{T} \rightarrow \text{Func}((\mathcal{T}^c)^\circ, \text{Vect}_k)$$

The previous lemma states that any functor that is cohomological and whose essential image lies in  $\text{vect}_k$  is in the essential image of the restricted Yoneda functor. Moreover, we know that the restricted Yoneda functor is full onto cohomological, finite-dimensional functors. The next obvious question concerns faithfulness.

**Definition 3.5.** *A morphism that lies in the kernel of the restricted Yoneda functor is called a **phantom map**.*

**Lemma 3.6.** *Let  $M$  be a bounded above complex of coherent sheaves on  $X$ . Then, there are no phantom maps from  $M$  to a bounded below complex.*

*Proof.* We can take a bounded above locally-free coherent resolution  $F$  of  $M$  and let  $F_{\geq i}$  denote the brutal truncation of  $F$  at the negative  $i$ -th step. This is the complex obtained from  $F$  by zeroing out the components for  $j > i$ . Each  $F_{\geq i}$  lies in  $D_{\text{perf}}(X)$ . Any map from  $M$  to a bounded below complex can be represented by an honest chain map from  $F$  to a bounded below complex of injectives  $I$ .

$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & F_{i-1} & \longrightarrow & F_i & \longrightarrow & F_{i+1} & \longrightarrow & \cdots & \longrightarrow & F_{N-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow 0 & & \downarrow 0 & & \downarrow & & & & \downarrow & & \downarrow 0 & & \downarrow 0 & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & I_{i+1} & \longrightarrow & \cdots & \longrightarrow & I_{N-1} & \longrightarrow & I_N & \longrightarrow & I_{N+1} & \longrightarrow & \cdots \end{array}$$

Since one complex is bounded above and the other is bounded below, the map must factor through some  $F_{\geq i}$ . The restriction to  $F_{\geq i}$  is null-homotopic if and only if the original map is null-homotopic since any homotopy must also factor through  $F_{\geq i}$ .

$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & F_{i-1} & \longrightarrow & F_i & \longrightarrow & F_{i+1} & \longrightarrow & \cdots & \longrightarrow & F_{N-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & \searrow 0 & \searrow 0 & \searrow 0 & \searrow & \searrow & \searrow 0 & \searrow 0 & \searrow 0 & & \searrow 0 & \searrow 0 & \searrow 0 & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & I_{i+1} & \longrightarrow & \cdots & \longrightarrow & I_{N-1} & \longrightarrow & I_N & \longrightarrow & I_{N+1} & \longrightarrow & \cdots \end{array}$$

Thus, the induced map from  $F_{< i}$  is zero if and only if the original map is zero.  $\square$

The proof says something stronger.

**Lemma 3.7.** *Take a bounded above complex of coherent sheaves  $M$  and a bounded below complex of quasi-coherent sheaves  $N$ . There exists a perfect object  $E$  and morphism  $E \rightarrow M$  that induces an isomorphism,  $[M, N] \cong [E, N]$ , of morphism spaces.*

The next step is to identify which objects of  $D(X)$  give rise to locally-finite functors.

**Lemma 3.8.**  $D_{\text{coh},c}^b(X)$  essentially surjects onto the category of locally-finite cohomological functors via the restricted Yoneda functor.

*Proof.* This is essentially from the appendix of [5]. First, consider the case of  $\mathbf{P}_k^N$ . Let  $\phi$  be a locally-finite cohomological functor on  $D_{\text{perf}}(\mathbf{P}_k^N) \cong D_{\text{coh}}^b(\mathbf{P}_k^N)$ . It is represented by a complex  $N$ . There is an equivalence between  $D(\text{Mod } A)$  and  $D_{\text{coh}}^b(\mathbf{P}_k^N)$ , which restricts to an equivalence between  $D_{\text{coh}}^b(\mathbf{P}_k^N)$  and  $D^b(\text{mod } A)$ .  $A$  is a finite dimensional algebra with finite global dimension. Identify  $N$  with its image. Since  $[A, N[j]] \cong H^j(N)$  we see that  $N$  has bounded finite-dimensional cohomology and, thus, lies in  $D^b(\text{mod } A)$ . So  $\phi$  is represented by an object of  $D_{\text{coh}}^b(\mathbf{P}_k^N)$ .

Now consider a locally-finite cohomological functor  $\phi$  on  $D_{\text{perf}}(X)$ .  $\phi$  itself is representable by complex  $M$  of  $D(X)$ . Choose an embedding  $i : X \hookrightarrow \mathbf{P}_k^N$  and consider  $\phi' = \phi \circ i^*$ .

$$\phi'(E) \cong [i^*E, M] \cong [E, i_*M]$$

Thus,  $i_*M$  represents  $\phi'$  and must lie in  $D_{\text{coh}}^b(\mathbf{P}_k^N)$ . Therefore,  $M$  lies in  $D_{\text{coh},c}^b(X)$ .  $\square$

Combining these lemmas gives the proof of theorem 3.3.  $\square$

**Remark 3.9.** The conclusion of theorem 3.3 also holds if  $X$  is Noetherian, possesses the resolution property, and  $D(X)$  satisfies an appropriate generation property. In [12], there is another proof of lemma 3.8 using this generation property. See proposition 6.12.

Staring at the local finiteness condition, one notices that any perfect object will furnish a locally-finite, homological functor on  $D_{\text{coh},c}^b(X)$ . Are these all of them? For a large class of schemes, this question is an easy corollary of results of Rouquier. The following theorem is quite powerful.

**Theorem 3.10.** (Rouquier [12]) Let  $X$  be a projective scheme over a perfect field  $k$ . Any cohomological or homological locally-finite functor on  $D_{\text{coh}}^b(X)$  is representable by an object  $D_{\text{coh}}^b(X)$ .

**Lemma 3.11.** Let  $X$  be a quasi-projective scheme over a field  $k$ . An object  $A \in D_{\text{coh}}^b(X)$  furnishes a locally-finite functor  $[A, -]$  on  $D_{\text{coh},c}^b(X)$  if and only if  $A$  is perfect.

*Proof.* As noted, if  $A$  is perfect, then  $[A, -]$  is locally-finite. Assume that  $[A, -]$  is locally-finite. Let  $x$  be closed point of  $X$ ,  $\mathcal{O}_{x,X}$  its local ring, and  $\mathcal{O}_x$  the structure sheaf of  $x$  in  $X$ . A bounded complex of finitely-generated  $\mathcal{O}_{x,X}$ -modules  $C$  is quasi-isomorphic to a bounded complex of free modules if and only if

$$\sum_{i \in \mathbf{Z}} \dim_k [C, \mathcal{O}_x[i]] < \infty.$$

Thus,  $A \otimes \mathcal{O}_{x,X}$  is quasi-isomorphic to a bounded complex of free  $\mathcal{O}_{x,X}$ -modules. This quasi-isomorphism extends to some open neighborhood of  $x$ . Thus,  $A$  is perfect.  $\square$

These results imply the following proposition.

**Proposition 3.12.** Let  $X$  be a projective scheme over a perfect field  $k$ . The inclusion  $D_{\text{perf}}(X) \rightarrow {}^\vee D_{\text{coh}}^b(X)$  is an equivalence.

**Remark 3.13.** *The natural question that now arises is: what happens if  $X$  is only quasi-projective? Breaking it down into sub-questions, one wonders: is any locally-finite homological functor  $\phi : D_{\text{coh},c}^b(X) \rightarrow \text{vect}_k$  represented by a perfect complex? Are there any nonzero morphisms between perfect complexes that induce the zero natural transformation as functors on  $D_{\text{coh},c}^b(X)$ ?*

#### 4. PSEUDO-ADJOINTS

In this section, unless otherwise stated,  $X$  and  $Y$  will be projective schemes over a field  $k$ .

It is important to keep in mind, when contemplating the results of the previous section, that there are two components of representability. The first is representing a functor by an object of an appropriate category. The second is representing natural transformations as morphisms between the representing objects. In the case that the appropriate category is actually the underlying category, the Yoneda lemma makes quick work of the second issue. However, in the cases of interest in this paper, the appropriate category is not the underlying category. Thus, the second issue must be addressed.

It is the representability of natural transformations that is the engine behind most results related to representability in the usual setting. For instance, general theorems giving sufficient conditions for representability of a functor have easy corollaries involving the existence of adjoints. In this section, we will collect some easy corollaries, many involving (pseudo-)adjunction, which follow from the representability results in the previous section.

**Definition 4.1.** *If  $F : D_{\text{perf}}(X) \rightarrow D_{\text{perf}}(Y)$  is an exact functor, a **right pseudo-adjoint** to  $F$  is a functor  $F^\vee : D_{\text{coh}}^b(Y) \rightarrow D_{\text{coh}}^b(X)$  so that we have natural isomorphisms*

$$[F(A), B] \cong [A, F^\vee(B)]$$

*for any pair of objects  $A$  in  $D_{\text{perf}}(X)$  and  $B$  in  $D_{\text{coh}}^b(Y)$ .*

*If  $G : D_{\text{coh}}^b(X) \rightarrow D_{\text{coh}}^b(Y)$  is an exact functor, a **left pseudo-adjoint** to  $G$  is a functor  ${}^\vee G : D_{\text{perf}}(Y) \rightarrow D_{\text{perf}}(X)$  so that we have natural isomorphisms*

$$[A, G(B)] \cong [{}^\vee G(A), B]$$

*for any pair of objects  $A$  in  $D_{\text{coh}}^b(X)$  and  $B$  in  $D_{\text{perf}}(Y)$ .*

**Remark 4.2.** *We can also extend the definition appropriately to the case where  $X$  and  $Y$  are quasi-projective. Hopefully, it will be clear which results in this section extend, after appropriate modification, to quasi-projective schemes over  $k$ .*

**Proposition 4.3.** *Any exact functor  $F : D_{\text{perf}}(X) \rightarrow D_{\text{perf}}(Y)$  possesses a right pseudo-adjoint.  $F^\vee$  is unique up to a unique isomorphism.*

*Proof.* By precomposition,  $F$  induces a functor  $D_{\text{perf}}(Y)^\vee \rightarrow D_{\text{perf}}(X)^\vee$ . Under the equivalence in theorem 3.3, this uniquely specifies  $F^\vee$ .  $\square$

Using the same method as the previous proof, and proposition 3.12 instead of theorem 3.3, we get a proof of the following.

**Proposition 4.4.** *If  $X$  and  $Y$  are projective schemes over a perfect field, then any exact functor  $G : D_{\text{coh}}^b(X) \rightarrow D_{\text{coh}}^b(Y)$  possesses a left pseudo-adjoint.*

**Lemma 4.5.** *If it exists,  ${}^\vee G$  is unique up to a unique isomorphism.*

*Proof.*  ${}^\vee G(A)$  represents the functor  $[A, G(-)]$  so it is unique up to a unique isomorphism. These isomorphisms are natural since the isomorphisms  $[{}^\vee G(A), B] \cong [A, G(B)]$  are natural.  $\square$

**Corollary 4.6.** *If  ${}^\vee G$  and  ${}^\vee F$  exist, we have natural isomorphisms*

$$\begin{aligned} {}^\vee(F^\vee) &\cong F \\ ({}^\vee G)^\vee &\cong G. \end{aligned}$$

*If  $F$  is an equivalence, then so is  $F^\vee$  and  $F$  admits an extension to  $D_{\text{coh}}^b(X)$ . If  $G$  is an equivalence, then so is  ${}^\vee G$  and  $G$  takes perfect objects to perfect objects.*

We can strengthen the result on equivalences somewhat slightly.

**Lemma 4.7.** *If  $G : D_{\text{coh}}^b(X) \rightarrow D_{\text{coh}}^b(Y)$  is an exact equivalence, then  $G$  restricts to an exact equivalence between  $D_{\text{perf}}(X) \rightarrow D_{\text{perf}}(Y)$ .*

*Proof.* From lemma 3.11, we know that, if  $[A, -]$  is a locally-finite homological functor with  $A \in D_{\text{coh}}^b(X)$ , then  $A$  lies in  $D_{\text{perf}}(X)$ . Since  $F$  is an equivalence,  $[A, -]$  is locally-finite if and only if  $[G(A), -]$  is locally-finite. Thus,  $G$  takes  $D_{\text{perf}}(X)$  to  $D_{\text{perf}}(Y)$ . Exactness follows from lemma 4.11.  $\square$

**Corollary 4.8.** *Let  $X$  a projective scheme over a field  $k$ . Then the groups of auto-equivalences of  $D_{\text{perf}}(X)$  coincides with the group of auto-equivalences of  $D_{\text{coh}}^b(X)$ .*

One can restrict an auto-equivalence of  $D(X)$  to get an auto-equivalence of  $D_{\text{perf}}(X)$ . There is an inverse to this restriction [1].

We have the following notion due to Orlov [11].

**Definition 4.9.** *Let  $X$  be a projective scheme over a field. The Verdier quotient*

$$D_{\text{coh}}^b(X)/D_{\text{perf}}(X)$$

*is called the **triangulated category of singularities** of  $X$  and is denoted by  $D_{\text{sing}}(X)$ .*

**Corollary 4.10.** *If  $X$  and  $Y$  are projective schemes over a field with equivalent bounded derived categories of coherent sheaves or perfect derived categories, then they have equivalent categories of singularities. In particular,  $X$  is regular if and only if  $Y$  is regular.*

**Lemma 4.11.** *If  $F : D_{\text{perf}}(X) \rightarrow D_{\text{perf}}(Y)$  is an exact functor, then  $F^\vee$  is also exact.*

*Proof.* We can define the shift functor on  $D_{\text{coh}}^b(X)$  (and  $D_{\text{coh}}^b(Y)$ ) purely in terms of locally-finite cohomological functors as the functor that sends  $\Phi$  to  $\Phi \circ [-1]$ .  $F^\vee$  is precomposition with  $F$  and thus commutes with shifts since  $F$  is exact.

The triangulated structure on  $D_{\text{coh}}^b(X)$  (and  $D_{\text{coh}}^b(Y)$ ) is not so easy to capture in terms of locally-finite cohomological functors. Indeed, this is the most common manifestation of the non-functoriality of the existence of cones in triangulated categories.

Take an exact triangle



$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
& \searrow [1] & \swarrow \\
& & C
\end{array}$$

in  $D_{\text{coh}}^b(Y)$ . Let  $Z$  be a cone over  $F^\vee(\alpha)$ . We seek a morphism  $\lambda : Z \rightarrow F^\vee(C)$  making

$$\begin{array}{ccccccc}
F^\vee(A) & \xrightarrow{F^\vee(\alpha)} & F^\vee(B) & \longrightarrow & Z & \longrightarrow & F^\vee(A)[1] \\
\downarrow = & & \downarrow = & & \downarrow \lambda & & \downarrow = \\
F^\vee(A) & \xrightarrow{F^\vee(\alpha)} & F^\vee(B) & \longrightarrow & F^\vee(C) & \longrightarrow & F^\vee(A)[1]
\end{array}$$

commute. Let  $E_i^{F^\vee(A)}$  denote the brutal truncation of a locally-free resolution of  $F^\vee(A)$  at the  $-i$ th stage and let  $E_j^{F^\vee(B)}$  denote the brutal truncation of a locally-free resolution of  $F^\vee(B)$  at the  $-j$ th stage. Since  $E_i^{F^\vee(A)}$  is compact and  $F^\vee(B)$  is the homotopy colimit of the  $E_j^{F^\vee(B)}$ , for large  $j$  there exists a map  $[F^\vee(\alpha)]_{ij}$  making

$$\begin{array}{ccc}
E_i^{F^\vee(A)} & \xrightarrow{[F^\vee(\alpha)]_{ij}} & E_j^{F^\vee(B)} \\
\downarrow & & \downarrow \\
F^\vee(A) & \xrightarrow{F^\vee(\alpha)} & F^\vee(B)
\end{array}$$

commute. Complete this to a diagram

$$\begin{array}{ccccccc}
E_i^{F^\vee(A)} & \xrightarrow{[F^\vee(\alpha)]_{ij}} & E_j^{F^\vee(B)} & \longrightarrow & C_{ij} & \longrightarrow & E_i^{F^\vee(A)}[1] \\
\downarrow & & \downarrow & & & & \downarrow \\
F^\vee(A) & \xrightarrow{F^\vee(\alpha)} & F^\vee(B) & \longrightarrow & F^\vee(C) & \longrightarrow & F^\vee(A)[1]
\end{array}$$

Let  $V$  be any bounded complex of coherent sheaves and make a choice of locally-free resolution of  $V$ , whose brutal truncation at the  $-i$ th step will be denoted by  $E_i$ . Given another bounded complex of locally-free coherent sheaves  $W$ , there exists an  $N_0$  such that for  $i > N_0$  the map  $E_i \rightarrow V$  induces an isomorphism

$$[V, W] \xrightarrow{\sim} [E_i, W]$$

of the morphism vector spaces. Consequently, for any  $V$ , there is morphism

$$FE_i^{F^\vee(V)} \rightarrow V$$

which corresponds to the identity under the isomorphisms

$$[F^\vee(V), F^\vee(V)] \cong [E_i^{F^\vee(V)}, F^\vee(V)] \cong [FE_i^{F^\vee(V)}, V].$$

These maps give a morphism of triangles

$$\begin{array}{ccccccc}
 F(E_i^{F^\vee(A)}) & \xrightarrow{F([F^\vee \alpha]_{ij})} & F(E_j^{F^\vee(B)}) & \longrightarrow & F(C_{ij}) & \longrightarrow & F(E_i^{F^\vee(A)})[1] \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A & \xrightarrow{\alpha} & B & \longrightarrow & C & \longrightarrow & A[1]
 \end{array}$$

from which we extract via adjunction maps  $C_{ij} \rightarrow F^\vee C$  for large  $i$  and  $j$ . Using the five lemma, we see that for any bounded complex of coherent sheaves  $V$  there are large  $i, j$  so that the map  $C_{ij} \rightarrow Z$  induces an isomorphism

$$[Z, V] \cong [C_{ij}, V].$$

Thus, we have the desired map  $\lambda : Z \rightarrow F^\vee(C)$ . To show that  $\lambda$  is an isomorphism, it is sufficient to show that it induces an isomorphism after application of  $[E, -]$  for each perfect  $E$ . This is done via adjunction and the five lemma similar to [3].  $\square$

The proof in [3] that adjoints of exact functors are exact extends to prove the following.

**Lemma 4.12.** *If  $F : D_{\text{coh}}^b(X) \rightarrow D_{\text{coh}}^b(Y)$  is an exact functor, then  ${}^\vee F : D_{\text{perf}}(Y) \rightarrow D_{\text{perf}}(X)$  is exact.*

## 5. ROUQUIER FUNCTORS

In section 6, we wish to demonstrate the utility of the ideas and results found in the previous sections by extending the reconstruction result of Bondal and Orlov to Gorenstein projective varieties. Before we can do this, we need one more idea, that of a Rouquier functor.

Let us recall the definition of a Serre functor.

**Definition 5.1.** *Let  $\mathcal{C}$  be a  $k$ -linear category. A **weak Serre functor**  $S : \mathcal{C} \rightarrow \mathcal{C}$  is an autofunctor for which there are natural isomorphisms*

$$\eta_{A,B} : [B, S(A)] \rightarrow [A, B]^*$$

*for any pair of objects  $A$  and  $B$  of  $\mathcal{C}$ .  $S$  is a **Serre functor** if, in addition, it is an autoequivalence.*

**Remark 5.2.** *This definition is slightly different than the one that commonly appears in the literature. We have reversed the dualization. Of course, if the category has finite dimensional morphism spaces, the two definitions are equivalent.*

**Example 5.3.** *The canonical example for a Serre functor is the following: let  $X$  be a smooth projective variety over a field  $k$  and let  $\omega_X$  be the canonical bundle. Then, the Serre functor on  $D_{\text{coh}}^b(X)$  is  $-\otimes \omega_X[\dim X]$ . This follows from Serre duality.*

**Definition 5.4.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be  $k$ -linear categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$  a  $k$ -linear functor. A **Rouquier functor**  $R_F$  for  $F$  is a  $k$ -linear functor  $R_F : \mathcal{C} \rightarrow \mathcal{D}$  for which there are natural isomorphisms*

$$\eta_{A,B} : [B, R_F(A)] \rightarrow [F(A), B]^*$$

Following [7], we denote the induced pairing

$$\begin{aligned} [B, R_F(A)] \otimes [F(A), B] &\longrightarrow k \\ (f, g) &\longmapsto \langle g|f \rangle. \end{aligned}$$

Naturality of  $R_F$  is expressed in terms of the bracket as  $\langle f \circ h|g \rangle = \langle f|h \circ g \rangle$  and  $\langle f \circ h|g \rangle = \langle h|g \circ f \rangle$ .

The definition of a Rouquier should be viewed as an attempt to relativize the notion of Serre functor. Indeed, if we take  $F$  to be the identity functor,  $R_F$  is just a weak Serre functor. The definition is natural so we should expect most categorical notions to respect Rouquier functors.

**Lemma 5.5.** *A necessary and sufficient condition for the existence of  $F$  is the representability of the functors  $[F(A), -]^*$  for objects  $A$  of  $\mathcal{C}$ .*

*Proof.* Clearly, if  $R_F$  exists, then  $[F(A), -]^*$  is representable. Indeed, it is represented by  $R_F(A)$ . Assume that  $[F(A), -]^*$  is representable for all  $A$ . We set  $R_F(A)$  to be a choice of the representing object for  $[F(A), -]^*$ . If we have a morphism  $\alpha : A \rightarrow A'$ , this gives a natural transformation of functors  $F(\alpha) : [F(A), -]^* \rightarrow [F(A'), -]^*$  which corresponds to a natural transformation  $[-, R_F(A)] \rightarrow [-, R_F(A')]$  and, by the Yoneda lemma, a morphism  $R_F(\alpha) : R_F(A) \rightarrow R_F(A')$ .  $\square$

**Lemma 5.6.** *If  $\mathcal{C}$  has finite-dimensional morphism spaces and  $F$  is fully-faithful, then  $R_F$  is fully-faithful.*

*Proof.* We have natural isomorphisms

$$[R_F(A), R_F(B)] \cong [F(B), R_F(A)]^* \cong [F(A), F(B)]^{**} \cong [F(A), F(B)] \cong [A, B].$$

$\square$

There is one useful case where we can guarantee the existence of a Rouquier functor. The following result and lemma 5.10 were first observed in [12], which inspired the name, Rouquier functor.

**Lemma 5.7.** *Let  $\mathcal{T}$  be a compactly-generated triangulated category. Then, the Rouquier functor for the inclusion  $\mathcal{T}^c \hookrightarrow \mathcal{T}$  exists.*

*Proof.* By lemma 5.5, we just need to show that  $[A, -]^*$  is representable for each compact  $A$ .  $[A, -]^*$  takes coproducts to products and is, thus, representable by Brown representability.  $\square$

We denote the Rouquier functor in this case by  $R_{\mathcal{T}}$ .

The same proof actually says a little more.

**Lemma 5.8.** *Let  $F$  be a functor from  $\mathcal{S}$  to  $\mathcal{T}$ . If  $\mathcal{T}$  is compactly generated and the essential image of  $F$  lies in  $\mathcal{T}^c$ , then  $R_F$  exists.*

Again, we return to the geometric setting for the main example.

**Example 5.9.** *Let  $X$  be a quasi-projective scheme over a field  $k$ . Let  $f : X \rightarrow \text{Spec } k$  be the structure map and  $f^!$  the right adjoint to  $f_*$ . The Rouquier functor  $R_X$  for  $D_{\text{perf}}(X) \hookrightarrow D(X)$  is  $-\otimes f^! \mathcal{O}_{\text{Spec } k}$ . Let  $A$  be a perfect complex and  $B$  a complex of quasi-coherent sheaves on  $X$ .*

$$[A, B]^* \cong [\mathcal{O}_X, A^\vee \otimes B]^* = (f_*(A^\vee \otimes B))^* \cong [A^\vee \otimes B, f^! \mathcal{O}_{\text{Spec } k}] \cong [B, A \otimes f^! \mathcal{O}_{\text{Spec } k}]$$

*If  $X$  is projective, then  $[A, -]^*$  is a locally-finite cohomological functor. So  $R_X$  maps  $D_{\text{perf}}(X)$  into  $D_{\text{coh}}^b(X)$ .*

**Lemma 5.10.** *Let  $\mathcal{T}$  be a compactly-generated triangulated category. If  $\mathcal{T}^c$  has a weak Serre functor  $S$ , it is isomorphic to  $R_{\mathcal{T}}$ .*

*Proof.* For any compact  $A$  and  $B$ , we have  $[B, R_{\mathcal{T}}(A)] \cong [A, B]^* \cong [B, S(A)]$ . By setting  $B = S(A)$ , we get a morphism  $\nu_A : S(A) \rightarrow R_{\mathcal{T}}(A)$ . Take a cone over this morphism and denote it by  $C$ . Since  $[B, \nu_A]$  is an isomorphism for any compact  $B$ ,  $[B, C] = 0$  for all compact  $B$  and  $C$  is zero.  $\square$

**Corollary 5.11.** *Let  $X$  be a quasi-projective scheme over a field  $k$ . If  $D_{\text{perf}}(X)$  possesses a weak Serre functor, it is  $R_X$  and  $f^! \mathcal{O}_{\text{Spec } k}$  is perfect.*

Assume we have  $k$ -linear equivalences  $\Phi : \mathcal{C} \rightarrow \mathcal{C}'$  and  $\Psi : \mathcal{D} \rightarrow \mathcal{D}'$  and  $k$ -linear functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $F' : \mathcal{C}' \rightarrow \mathcal{D}'$  making the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow \Phi & & \downarrow \Psi \\ \mathcal{C}' & \xrightarrow{F'} & \mathcal{D}' \end{array}$$

commute.

**Lemma 5.12.** *If  $R_F$  exists, then so does  $R_{F'}$ . Moreover,  $R_{F'} \circ \Phi$  is naturally isomorphic to  $\Psi \circ R_F$ .*

*Proof.* Since  $[F(+), \Psi^{-1}(-)] \cong [\Psi \circ F(+), -] \cong [F' \circ \Phi(+), -]$ , we have natural isomorphisms  $[\Psi^{-1}(-), R_F(+)] \cong [-, \Psi \circ R_F(+)] \cong [-, R_{F'} \circ \Phi(+)]$ , and, consequently, a natural isomorphism  $\Psi \circ R_F \cong R_{F'} \circ \Phi$ .  $\square$

**Remark 5.13.** *We have seen an equivalence of either the perfect derived categories or the bounded derived categories of coherent sheaves of two projective schemes induces an equivalence of the other pair of categories. By lemma 5.12, we see that these equivalences must commute the Rouquier functors.*

**Proposition 5.14.** *If  $F$  is an exact functor between triangulated categories, then  $R_F$  is also exact.*

*Proof.* By lemma 5.12,  $R_F$  commutes with shift functors. To prove that  $R_F$  takes triangles to triangles, we follow the proof of the analogous claim for a Serre functor in [7]. We claim that the diagram

$$\begin{array}{ccc} [F(A), F(B)]^* & \xleftarrow{\eta_{A, F(B)}} & [F(B), R_F(A)] \\ \downarrow F^* & & \searrow \\ & & [F(B), R_F(A)]^{**} \\ & & \swarrow \eta_{B, R_F(A)}^* \\ [A, B]^* & \xleftarrow{R_F^*} & [R_F(A), R_F(B)]^* \end{array}$$

commutes. Extending the pairing  $\langle \cdot | \cdot \rangle$  by dualising (and using the natural isomorphism  $k^* \cong k$ ), we can rewrite the commutativity of the diagram as the equality of the natural pairing

$$\langle h | F(g) \rangle = \langle R_F(g) | h \rangle$$

for  $h \in [F(B), R_F(A)]$  and  $g \in [A, B]$ . Using the naturality of  $\eta$ , we have a commutative diagram

$$\begin{array}{ccc} [F(A), F(B)]^* & \xleftarrow{\eta_{A, F(B)}} & [F(B), R_F(A)] \\ \downarrow F^*(g) & & \downarrow R_F(g) \\ [F(B), F(B)]^* & \xleftarrow{\eta_{B, F(B)}} & [F(B), R_F(B)] \end{array}$$

which translates into  $\langle h|F(g)\rangle = \langle R_F(g) \circ h|\text{id}_{F(B)}\rangle$ . Again using the naturality of  $\eta$ , we have another commutative diagram

$$\begin{array}{ccc} [F(B), F(B)]^* & \xleftarrow{\eta_{B, F(B)}} & [F(B), R_F(B)] \\ \uparrow h^* & & \uparrow h \\ [F(B), R_F(A)]^* & \xleftarrow{\eta_{B, R_F(A)}} & [R_F(A), R_F(B)] \end{array}$$

which gives  $\langle R_F(g) \circ h|\text{id}_{F(B)}\rangle \cong \langle R_F(g)|h\rangle$  and proves the claim.

Let

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ & \searrow \gamma & \swarrow \beta \\ & C & \end{array}$$

be an exact triangle and let  $C_0$  be an object which completes the morphism  $R_F(\alpha) : R_F(A) \rightarrow R_F(B)$  to an exact triangle. We now seek a morphism  $\xi : C_0 \rightarrow R_F(C)$  which makes

$$\begin{array}{ccccccc} R_F(A) & \xrightarrow{R_F(\alpha)} & R_F(B) & \xrightarrow{\phi} & C_0 & \xrightarrow{\psi} & R_F(A)[1] \\ \downarrow = & & \downarrow = & & \downarrow \xi & & \downarrow = \\ R_F(A) & \xrightarrow{R_F(\alpha)} & R_F(B) & \xrightarrow{R_F(\beta)} & R_F(C) & \xrightarrow{R_F(\gamma)} & R_F(A)[1] \end{array}$$

commute. Assume we have such a morphism. Since the bottom diagram is exact after application of  $[D, -]$  for any  $D$ , we see that  $[D, \xi]$  induces an isomorphism for all  $D$ . Consequently,  $\xi$  is an isomorphism.

Finding such a  $\xi$  is equivalent to finding a linear map  $\langle \xi| : [F(C), C_0] \rightarrow k$  satisfying

$$\langle \xi \circ \phi|\lambda\rangle = \langle R_F(\beta)|\lambda\rangle$$

for all  $\lambda \in [F(C), R_F(B)]$  and

$$\langle R_F(\gamma) \circ \xi|\mu\rangle = \langle \psi|\mu\rangle$$

for all  $\mu \in [FA[1], C_0]$ . Using the functoriality of the pairing and the claim, we reduce to requiring that

$$\langle \xi | \phi \circ \rangle = \langle \text{id}_{F(B)} | \lambda \circ F(\beta) \rangle \text{ and } \langle \xi | \circ F(\gamma) \rangle = \langle \text{id}_{F(A)[1]} | \psi \circ \mu \rangle$$

on  $[F(C), R_F(B)]$  and on  $[F(A)[1], C_0]$  respectively. To guarantee that  $\langle \xi |$ , and consequently  $\xi$ , exists, we need

$$\langle \text{id}_{F(B)} | \lambda \circ F(\beta) \rangle = \langle \text{id}_{F(A)[1]} | \psi \circ \mu \rangle$$

if  $\phi \circ \lambda = \mu \circ F(\gamma)$ . We can complete to a morphisms of triangles

$$\begin{array}{ccccccc} F(B) & \xrightarrow{F(\beta)} & F(C) & \xrightarrow{F(\gamma)} & F(A)[1] & \xrightarrow{F(\alpha)} & F(B)[1] \\ \downarrow \nu & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu[1] \\ R_F(A) & \xrightarrow{R_F(\alpha)} & R_F(B) & \xrightarrow{\phi} & C_0 & \xrightarrow{\psi} & R_F(A)[1] \end{array}$$

which gives the relations  $\lambda \circ F(\beta) = R_F(\alpha) \circ \nu$  and  $\psi \circ \mu = \nu[1] \circ F(\alpha)[1]$ . Using the naturality of  $\eta$ , commutation with shifts, and the claim, we have the following equalities

$$\begin{aligned} \langle \text{id}_{F(B)} | \lambda \circ F(\beta) \rangle &= \langle \text{id}_{F(B)} | R_F(\alpha) \circ \nu \rangle = \langle \nu | F(\alpha) \rangle = \\ &= \langle \text{id}_{F(A)[1]} | \nu[1] \circ F(\alpha)[1] \rangle = \langle \text{id}_{F(A)[1]} | \psi \circ \mu \rangle \end{aligned}$$

□

## 6. AN APPLICATION: RECONSTRUCTION FOR PROJECTIVE GORENSTEIN VARIETIES

For this section, a variety is a reduced and irreducible scheme of finite type over a field  $k$ . In [4], Bondal and Orlov prove the following reconstruction result.

**Theorem 6.1.** *Let  $X$  be a smooth projective variety over a field  $k$  with ample or anti-ample canonical bundle. Assume there is another smooth variety,  $Y$ , and an exact equivalence  $D_{\text{coh}}^b(X) \cong D_{\text{coh}}^b(Y)$ . Then,  $X$  is isomorphic to  $Y$ .*

In this section, we use the results of the previous sections to carry forth the original argument of Bondal and Orlov and prove the following extension.

**Proposition 6.2.** *Let  $X$  be a projective Gorenstein variety over a field  $k$  with ample or anti-ample canonical bundle. Assume that  $Y$  is a projective variety over  $k$  and there is an exact equivalence between  $D_{\text{perf}}(X)$  and  $D_{\text{perf}}(Y)$ . Then,  $X$  is isomorphic to  $Y$ .*

An immediate corollary to proposition 6.2, thanks to lemma 4.7, is the following.

**Corollary 6.3.** *Let  $X$  be a projective Gorenstein variety over a field  $k$  with ample or anti-ample canonical bundle. Assume that  $Y$  is a projective variety over  $k$  and there is an exact equivalence between  $D_{\text{coh}}^b(X)$  and  $D_{\text{coh}}^b(Y)$ . Then,  $X$  is isomorphic to  $Y$ .*

**Definition 6.4.** *A scheme  $X$  over a field  $k$  is **Gorenstein** if  $D_{\text{perf}}(X)$  has a Serre functor. If  $X$  is a projective variety, this is equivalent to  $f^! \mathcal{O}_{\text{Spec } k}$  being a shift of a line bundle.*

**Lemma 6.5.** *If  $X$  is projective and Gorenstein, then  $f^! \mathcal{O}_X$  is quasi-isomorphic to a line bundle concentrated in  $-\dim X$ .*

*Proof.* In this case the dualizing sheaf has an explicit construction as  $\mathcal{E}xt^r(\mathcal{O}_X, \omega_{\mathbf{P}_k^N})$  where  $X \hookrightarrow \mathbf{P}_k^N$  is an embedding and  $r$  is the codimension of  $X$  in  $\mathbf{P}_k^N$ .  $\square$

If  $X$  is Gorenstein, we have a Serre functor  $S$  on  $D_{\text{perf}}(X)$ .  $S^\vee$ , the right pseudo-adjoint to  $S$ , induces an autoequivalence  $D_{\text{coh}}^b(X) \rightarrow D_{\text{coh}}^b(X)$ , which must be  $-\otimes \omega_X^{-1}[-\dim X]$ . We shall use this to characterize points a'la [4].

**Definition 6.6.** *An **point functor**  $P$  of codimension  $d$  of  $D_{\text{perf}}(X)$  is a locally-finite cohomological functor on  $D_{\text{perf}}(X)$ , which satisfies the following conditions*

- (1)  $S^\vee(P) \cong P[-d]$ .
- (2)  $[P, P[l]]$  is zero for  $l < 0$ .
- (3)  $[P, P] \cong k(P)$  a finite field extension of  $k$ .

**Lemma 6.7.** *Let  $X$  be a projective Gorenstein variety. Any point object must have codimension  $\dim X$ .*

*Proof.* From our assumptions, we can apply locally-finite duality. By uniqueness,

$$S^\vee \cong \omega_X^{-1}[-\dim X] \otimes -$$

By theorem 3.3,  $P$  is represented by a bounded complex of coherent sheaves which we also denote by  $P$ . Let  $P$  have codimension  $d$ . Since  $S^\vee(P) \cong P[-d]$ , we know that  $\omega_X^{-1} \otimes P$  is quasi-isomorphic to  $P[-d + \dim X]$ . Let  $\mathcal{H}^i$  denote the cohomology sheaves of  $P$ . Since  $P$  has bounded cohomology and  $\omega_X \otimes \mathcal{H}^i \cong \mathcal{H}^{i-d+\dim X}$ , either  $P$  is quasi-isomorphic to zero (and is not a point object) or  $d = \dim X$ .  $\square$

**Lemma 6.8.** *Let  $X$  be a projective Gorenstein variety over a field  $k$  with ample or anti-ample canonical bundle. Then, an object of  $P$  of  $D_{\text{coh}}^b(X)$  is a point object if and only if  $P$  is isomorphic to  $\mathcal{O}_p[r]$  for some closed point  $p$  of  $X$ .*

*Proof.* We shall follow the proof in [7]. Note that shifts of points are point objects of codimension  $\dim X$ . Since  $S^\vee(P) \cong P[-\dim X]$ , we know that  $\omega_X^{-1} \otimes P$  is quasi-isomorphic to  $P$ . Let  $\mathcal{H}^i$  denote the cohomology sheaves of  $P$ . Because  $\omega_X$  is ample (or anti-ample) and  $\omega_X \otimes \mathcal{H}^i \cong \mathcal{H}^i$ ,  $\mathcal{H}^i$  has zero dimensional support.

We can resolve  $P$  using direct sums of injective sheaves each of whose support is contained within an irreducible component of the support of the cohomology sheaves of  $P$ . Since any map between two quasi-coherent sheaves with disjoint support is zero, we get a splitting of  $P$  into complexes supported at single points. Since  $[P, P]$  is a field, all summands but one must be quasi-isomorphic to zero.

Assume  $P$  has cohomology supported only a single point of  $X$ . Let  $m_0$  be the minimal  $i$  so that  $\mathcal{H}^i$  is nonzero and  $m_1$  the maximal  $i$  so that  $\mathcal{H}^i$  is nonzero. By truncating, we can assume that  $P$  is zero outside  $[m_0, m_1]$ . And there are morphisms  $\mathcal{H}^{m_0} \rightarrow P[m_0]$  and  $P[m_1] \rightarrow \mathcal{H}^{m_1}$ . For each  $\mathcal{H}^{m_0}$  and  $\mathcal{H}^{m_1}$ , there are nonzero maps in and out of  $k(x)$ . Composing these give a nontrivial element of  $[P[m_0], P[m_1]]$ . Thus,  $m_0$  equals  $m_1$  and  $P$  is a shift of a coherent sheaf. If the length of  $P$  is greater than one we can project down the composition series to get a non-invertible map. Thus,  $P$  is simply  $\mathcal{O}_p[r]$ .  $\square$

**Definition 6.9.** *An object  $L$  of  $D_{\text{perf}}(X)$  is an **locally-free object** if there exists a  $t \in \mathbf{Z}$  and  $n > 0$  so that for any point object  $P$  of codimension  $\dim X$*

- (1)  $\text{Hom}_T(L, P[t]) \cong k(P)^n$  and

(2)  $\mathrm{Hom}_{\mathcal{T}}(L, P[i]) = 0$  for  $i \neq t$ .

**Lemma 6.10.** *Let  $X$  be a projective variety. A perfect object,  $L$ , satisfies the two conditions of definition 6.9 for all shifts of closed points if and only if  $L$  is isomorphic to a shift of a locally-free sheaf. In particular, if all the point objects of  $X$  are isomorphic to shifts of points, then  $L$  is an locally-free object if and only if  $L$  is a shift of a locally-free coherent sheaf.*

*Proof.* We can replace  $L$  with a bounded complex of locally-free coherent sheaves, also denoted by  $L$  and compute  $[L, \mathcal{O}_p[l]]$  as  $H^l(\check{C}(\mathcal{U}, L^\vee \otimes \mathcal{O}_p))$ . We need a lemma.

**Lemma 6.11.** *Let  $X$  be a reduced quasi-projective scheme over a field  $k$ . Let  $P$  be a bounded complex of locally-free coherent sheaves. If  $P \otimes_{\mathcal{O}_p}$  is quasi-isomorphic to  $k(p)^n$  (in degree zero) for all closed points  $p$  in  $X$ . Then,  $P$  is quasi-isomorphic to a locally-free sheaf of rank  $n$ .*

*Proof.* Since  $X$  is reduced and the closed points are dense in  $X$ , for any nonzero coherent sheaf  $H$ ,  $H \otimes_{\mathcal{O}_p}$  must be nonzero for some closed point  $p \in X$ . Consider the cohomology sheaves  $\mathcal{H}^i$  of  $P$ . If  $\mathcal{H}^i \otimes_{\mathcal{O}_p}$  is nonzero, it must contribute to the  $i$ th cohomology of  $P \otimes_{\mathcal{O}_p}$ . Thus, we see that  $P$  is concentrated in degree zero.  $P$  is therefore quasi-isomorphic to a coherent sheaf, which admits a bounded locally-free resolution. One simply truncates to give the following:

$$0 \rightarrow P^{-i} \rightarrow \dots \rightarrow P^{-1} \rightarrow \ker d_0 \rightarrow \mathcal{H}^0 \rightarrow 0.$$

$\ker d_0$  is locally-free since it admits a bounded above (right) locally-free resolution. So, without loss of generality, we can replace  $P$  by a bounded complex of locally-free coherent sheaves that is exact except at the right end. Let  $P_0$  denote  $\ker d_0$ . We want to compute  $P^0 \otimes_{\mathcal{O}_p} / \mathrm{im}(P^{-1} \otimes_{\mathcal{O}_p})$ . But,  $- \otimes_{\mathcal{O}_p}$  is right exact so  $P^0 \otimes_{\mathcal{O}_p} / \mathrm{im}(P^{-1} \otimes_{\mathcal{O}_p})$  is just  $\mathcal{H}^0 \otimes_{\mathcal{O}_p}$ . Since  $\mathcal{H}^0$  has constant rank at all closed points in  $X$ , it must be locally-free of rank  $n$ .  $\square$

Since  $L^\vee$  is quasi-isomorphic to an locally-free coherent sheaf, so is  $L$ .  $\square$

Now we move onto the proof of proposition 6.2. It will be accomplished through a sequence of lemmas. Assume that  $X$  and  $Y$  satisfy the hypotheses of proposition 6.2.  $D$  will be a placeholder for  $D_{\mathrm{perf}}(X)$  or  $D_{\mathrm{perf}}(Y)$  depending on the context. Since they are equivalent, this should cause no confusion.

**Lemma 6.12.** *All point functors of  $Y$  are shifts of points.*

*Proof.* Let us denote the category of point objects of  $Z$  as  $\mathcal{P}(Z)$ . From lemma 6.8, we know there is a bijection between shifts of closed points of  $X$  and objects of  $\mathcal{P}(X)$ . From our assumption we have an equivalence between  $\mathcal{P}(X)$  and  $\mathcal{P}(Y)$  and there is inclusion of the shifts of points on  $Y$  into  $\mathcal{P}(Y)$ . The category  $\mathcal{P}(X)$  satisfies the following condition: if  $P$  and  $Q$  are objects, then either  $P$  is isomorphic to a shift of  $Q$  or  $[P, Q[j]]$  is zero for all integers  $j$ . If  $N$  is nonzero, then let  $m$  be the largest integer for which the  $m$ -th cohomology sheaf is nonzero. There must be a nonzero map in  $[N, \mathcal{O}_y[-m]]$ . Thus,  $N$  must be a shift of a point. So all point objects of  $Y$  are shifts of points.  $\square$

We can then apply lemma 6.10.

**Lemma 6.13.** *All locally-free objects of  $D_{\mathrm{perf}}(Y)$  are shifts of locally-free coherent sheaves.*



**Lemma 6.14.** *The underlying topological spaces of  $X$  and  $Y$  (as varieties) are homeomorphic.*

*Proof.* Choose an invertible object  $L_0$  corresponding to an invertible sheaf on  $X$ . By shifting, we can assume our equivalence takes  $L_0$  to a complex quasi-isomorphic to an invertible sheaf on  $Y$ . Let us denote the image by  $L_0$  also. Now the set of point objects  $P$  so that  $[L_0, P]$  is  $k(P)$  is in bijection with the set of closed points of  $X$ . Denote this set by  $p_D$ . Similarly,  $p_D$  is in bijection with the closed points of  $Y$ . This gives us a bijection between the closed points of  $X$  and  $Y$ .

Let  $l_D$  denote the set of locally-free objects  $L$  in  $D$  so that  $[L, P]$  is isomorphic to  $k(P)^n$  for some  $n$  and for all  $P$  in  $p_D$ .  $l_D$  is in bijection with the set of locally-free sheaves on  $X$  and the set of locally-free sheaves on  $Y$ . For  $\alpha$  in  $[L, L']$  with  $L, L' \in l_D$ , let  $U_\alpha$  denote the set of  $P$  in  $p_D$  so that the induced map

$$- \circ \alpha : [L', P] \rightarrow [L, P]$$

is zero. Since  $X$  and  $Y$  possess enough locally-free coherent sheaves, we know the open sets  $U_\alpha$  in  $X$  and  $Y$  form a basis for the topologies of  $X$  and  $Y$ , see [8]. Thus, our identification of points is a homeomorphism.  $\square$

Therefore, the dimensions of  $X$  and  $Y$  must coincide. Record this common dimension as  $d$ .

**Lemma 6.15.**  *$Y$  is Gorenstein and  $\omega_Y$  is ample.*

*Proof.* Being Gorenstein is categorical. Let  $L$  be a line bundle on an algebraic variety  $V$ .  $U_\alpha$  for  $\alpha$  in  $[L^{\otimes i}, L^{\otimes j}]$  form a basis for the topology of  $V$  if and only if  $L$  is ample [8]. We see that our equivalence takes ample invertible sheaves to ample invertible sheaves.

We can twist our equivalence by an invertible sheaf and assume that the structure sheaf of  $X$  is sent to the structure sheaf of  $Y$ . Then, from the naturality of the Serre functors,  $\omega_X[d]$  is sent to the dualising complex of  $Y$ . The dualising complex of  $Y$  must therefore be a shift of an invertible sheaf. Consequently,  $Y$  is Gorenstein.  $\square$

**Lemma 6.16.** *The graded rings  $\bigoplus_{n \in \mathbf{Z}} H^0(\mathcal{O}_X, \omega_X^n)$  and  $\bigoplus_{n \in \mathbf{Z}} H^0(\mathcal{O}_Y, \omega_Y^n)$  are isomorphic. Consequently,  $X$  and  $Y$  are isomorphic.*

*Proof.* Set  $L_i$  equal to  $S^i L_0[-di]$  where  $L_0$  is as chosen before. For each pair  $(i, j)$ , we have natural isomorphisms

$$[L_i, L_j] \cong [S^i L_0[-di], S^j L_0[-dj]] \cong [L_0, S^{j-i} L_0[-d(j-i)]] \cong [L_0, L_{j-i}]$$

This provides the structure of a graded ring for  $A = \bigoplus_{l=-\infty}^{\infty} \text{Hom}_D(L_0, L_l)$ . But,  $A$  is isomorphic to  $\bigoplus_{l \in \mathbf{Z}} H^0(X, \omega_X^l)$  and  $\bigoplus_{l \in \mathbf{Z}} H^0(Y, \omega_Y^l)$ . Since both  $\omega_X$  and  $\omega_Y$  are either ample or anti-ample, we can take  $\text{Proj}$  of the appropriate half to give  $X \cong Y$ .  $\square$

**Remark 6.17.** *Reconstruction can be accomplished with less hypotheses on  $X$  and  $Y$ . For more on this approach, see [2].*

One can also carry forth the arguments used in [4] to prove the following.

**Proposition 6.18.** *Let  $X$  be a projective Gorenstein variety with ample or anti-ample canonical bundle. Then, the group of auto-equivalences of  $D_{\text{perf}}(X)$  (and  $D_{\text{coh}}^b(X)$ ) is isomorphic to  $\text{Aut}(X) \rtimes (\text{Pic}(X) \times \mathbf{Z})$ .*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA 19104, USA

*E-mail address:* ballardm@math.upenn.edu